

THE YAMABE PROBLEM WITH SINGULARITIES

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ABSTRACT. Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Under some assumptions, we prove that there exists a positive function φ solution of the following Yamabe type equation

$$\Delta\varphi + h\varphi = \tilde{h}\varphi^{\frac{n+2}{n-2}}$$

where $h \in L^p(M)$, $p > n/2$ and $\tilde{h} \in \mathbb{R}$. We give the regularity of φ with respect to the value of p . Finally, we consider the results in geometry when g is a singular Riemannian metric and $h = \frac{n-2}{4(n-1)}R_g$, where R_g is the scalar curvature of g .

1. INTRODUCTION

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. Denote by R_g the scalar curvature of g . The Yamabe problem is the following:

Problem 1. Does there exists a constant scalar curvature metric conformal to g ?

If $\tilde{g} = \varphi^{4/(n-2)}g$ is a conformal metric to g with φ a smooth positive function, then the scalar curvatures R_g and $R_{\tilde{g}}$ are related by the following equation:

$$(1) \quad \frac{4(n-1)}{n-2}\Delta_g\varphi + R_g\varphi = R_{\tilde{g}}\varphi^{N-1}$$

where $N = \frac{2n}{n-2}$ and Δ_g is the geometric Laplacian of the metric g with nonnegative eigenvalues. To solve the Yamabe problem, it is equivalent to find a function φ solution of equation above where $R_{\tilde{g}}$ is constant. Equation (1) is called Yamabe equation. Yamabe [11] stated the following functional, defined for any $\psi \in H_1(M) - \{0\}$ by

$$(2) \quad I_g(\psi) = \frac{E(\psi)}{\|\psi\|_N^2} = \frac{\int_M |\nabla\psi|^2 + \frac{n-2}{4(n-1)}R_g\psi^2 dv}{\|\psi\|_N^2}$$

and he considered the infimum of I_g defined as follow

$$\mu(g) = \inf_{\psi \in H_1(M) - \{0\}} I_g(\psi)$$

He solved the case when $\mu(g)$ is nonpositive. Aubin [1] showed that it was sufficient to solve the following conjecture:

Conjecture 2 (Aubin [1]). *If (M, g) is not conformal to (S_n, g_{can}) then*

$$(3) \quad \mu(M, g) < \mu(S_n, g_{can})$$

where $\mu(M, g) = \inf\{I_g(\psi), \psi \in H_1(M) - \{0\}\}$

It is known that $\mu(S_n, g_{can}) = K^{-2}(n, 2) = \frac{1}{4}n(n-2)\omega_n^{2/n}$, where ω_n is the volume of the unit sphere S_n and $K(n, 2)$ is defined in theorem 7.

In the following, we write $\mu(g)$ instead of $\mu(M, g)$.

Aubin proved that the conjecture is valid for all smooth compact non conformally flat Riemannian manifolds of dimension $n \geq 6$ and conformally flat manifolds with finite non trivial fundamental group. The case of conformally flat manifolds and the dimensions 3, 4 and 5 were solved by Schoen [9] using positive mass theorem. Hence the conjecture above holds. By works of Yamabe [11], Aubin [1] and Schoen [9], the Yamabe problem is completely solved, when the

manifold is compact and smooth.

The purpose of this paper is to study the following equation

$$(4) \quad \Delta_g \psi + h\psi = \tilde{h}\psi^{\frac{n+2}{n-2}}$$

where $h \in L^p(M)$, and $\tilde{h} \in \mathbb{R}$. We call this kind of equation "Yamabe type equation". We will give a special consideration for the case $h = \frac{n-2}{4(n-1)}R_g$.

2. REGULARITY THEOREMS FOR YAMABE TYPE EQUATIONS

Theorem 3. *Let Ω be an open subset of \mathbb{R}^n and L an uniformly elliptic linear operator of the second degree, defined by*

$$(5) \quad L(u) = \sum_{i,j} a_{ij} \partial_{ij} u + \sum_i b_i \partial_i u + hu$$

where the coefficients a_{ij} , b_i and h are real valued bounded functions of class C^k with $k \in \mathbb{N}$. Let u be a weak solution of the equation $Lu = f$.

(i) *If $f \in C^{k,\alpha}(\Omega)$ then $u \in C^{k+2,\alpha}(\Omega)$*

(ii) *If $f \in H_k^p(\Omega)$ then $u \in H_{k+2}^p(\Omega)$*

This theorem is the standard regularity theorem, we can find a proof in the book of Gilbarg and Trudinger [7].

The following two theorems allow us to find the best regularity for the solution of Yamabe type equations. Using the theorem 4, Trudinger [10] Showed that the weak solutions of the Yamabe equation (1) are smooth. Yamabe [11] had already used implicitly this theorem.

Theorem 4. *On a n -dimensional compact Riemannian manifold (M, g) , if $u \geq 0$ is a non trivial weak solution in $H_1(M)$ of equation $\Delta_g u + hu = 0$, with $h \in L^p(M)$ and $p > n/2$, then $u \in C^{1-[n/p],\beta}(M)$ and positive. $[n/p]$ is the integer part of n/p , $\beta \in (0, 1)$.*

Notice that if u satisfies the assumptions of this theorem, then $\Delta u \in L^p(M)$. Regularity theorem 3 implies that $u \in H_2^p(M)$ and using Sobolev embedding, we find $u \in C^{1-[n/p],\beta}(M)$

Theorem 4 permits to proof the following theorem:

Theorem 5. *Let (M, g) be n -dimensional compact smooth Riemannian manifold. p and \tilde{h} are two reel numbers, with $p > n/2$. If $\varphi \in H_1(M)$ is a non trivial, nonnegative weak solution of*

$$(6) \quad \Delta_g \psi + h\psi = \tilde{h}\psi^{\frac{n+2}{n-2}}$$

then $\varphi \in H_2^p(M) \subset C^{1-[n/p],\beta}(M)$ and φ is positive.

Proof. It is sufficient to show that there exists $\varepsilon > 0$ such that $\varphi \in L^{(\varepsilon+2n)/(n-2)}(M)$. Indeed, if φ satisfies the assumptions of theorem and belongs to $L^{(\varepsilon+2n)/(n-2)}(M)$, then it is a solution of

$$\Delta_g u + (h - \tilde{h}\varphi^{\frac{4}{n-2}})u = 0$$

with $h - \tilde{h}\varphi^{\frac{4}{n-2}} \in L^r(M)$ and $r = \min(p, \frac{2n+\varepsilon}{4}) > n/2$. Using theorem 4, we deduce that φ is positive and continuous. Theorem 3 and Sobolev embedding imply that $\varphi \in H_2^p(M)$ with $p > n/2$.

Let l be a positive reel number and H, F are two continuous functions in \mathbb{R}_+ defined by:

$$H(t) = \begin{cases} t^\gamma & \text{if } 0 \leq t \leq l \\ l^{q-1}(ql^{q-1}t - (q-1)l^q) & \text{if } t > l \end{cases}$$

$$F(t) = \begin{cases} t^q & \text{if } 0 \leq t \leq l \\ ql^{q-1}t - (q-1)l^q & \text{if } t > l \end{cases}$$

$$(7) \quad \text{where } \gamma = 2q - 1, \text{ and } 1 < q < \frac{n(p-1)}{p(n-2)}$$

φ is positive, belongs to $H_1(M)$. $H \circ \varphi$ and $F \circ \varphi$ belong also to $H_1(M)$. Notice that for any $t \in \mathbb{R}_+ - \{l\}$

$$(8) \quad qH(t) = F(t)F'(t), (F'(t))^2 \leq qH'(t) \text{ and } F^2(t) \geq tH(t)$$

If φ is a weak solution of equation (6), then

$$(9) \quad \forall \psi \in H_1(M) \quad \int_M \nabla \varphi \cdot \nabla \psi dv + \int_M h \varphi \psi dv = \tilde{h} \int_M \varphi^{N-1} \psi dv$$

where $N = 2n/(n-2)$.

Let us choose $\psi = \eta^2 H \circ \varphi$, where η is C^1 -function with support in the ball $B_P(2\delta)$ and radius 2δ sufficiently small, such that $\eta = 1$ on $B_P(\delta)$. If we substitute in (9), we obtain

$$(10) \quad \int_M \eta^2 H' \circ \varphi |\nabla \varphi|^2 dv + 2 \int_M \eta H \circ \varphi \nabla \varphi \cdot \nabla \eta dv = \tilde{h} \int_M \varphi^{N-1} \eta^2 H \circ \varphi dv - \int_M h \varphi \eta^2 H \circ \varphi dv$$

Let $f = F \circ \varphi$ be a function. We estimate the forth integrals above, using function f and relations (8). We have $\nabla f = F' \circ \varphi \nabla \varphi$, the second relation in (8) implies

$$|\nabla f|^2 = (F' \circ \varphi)^2 |\nabla \varphi|^2 \leq q H' \circ \varphi |\nabla \varphi|^2$$

We deduce that the first integral of equality (10) is bounded from below.

$$\frac{1}{q} \|\eta \nabla f\|_2^2 \leq \int_M \eta^2 H' \circ \varphi |\nabla \varphi|^2 dv$$

The first relation of (8) and Cauchy-Schwarz inequality imply that the second integral of (10) is bounded from below by:

$$2 \int_M \eta H \circ \varphi \nabla \varphi \cdot \nabla \eta dv = \frac{2}{q} \int_M \eta f \nabla f \nabla \eta dv \geq \frac{-2}{q} \|f \nabla \eta\|_2 \|\eta \nabla f\|_2$$

By the last relation in (8), we have $\varphi H \circ \varphi \leq f^2$. The two integrals in the right side in (10) are bounded by:

$$\left| \tilde{h} \int_M \varphi^{N-1} \eta^2 H \circ \varphi dv - \int_M h \varphi \eta^2 H \circ \varphi dv \right| \leq |\tilde{h}| \|\varphi\|_{N,2\delta}^{4/(n-2)} \|\eta f\|_N^2 + \|h\|_p \|\eta f\|_{2p/(p-1)}^2$$

where $\|\varphi\|_{N,r}^N = \int_{B_P(r)} \varphi^N dv$. If we take together these estimates, equality (10) becomes:

$$(11) \quad \|\eta \nabla f\|_2^2 - 2 \|f \nabla \eta\|_2 \|\eta \nabla f\|_2 \leq q(|\tilde{h}| \|\varphi\|_{N,2\delta}^{4/(n-2)} \|\eta f\|_N^2 + \|h\|_p \|\eta f\|_{2p/(p-1)}^2)$$

Notice that for all nonnegative reel numbers a, b, c and d , if $a^2 - 2ab \leq c^2 + d^2$ then $a \leq c + d + 2b$. Using this remark, inequality (11) becomes:

$$(12) \quad \|\eta \nabla f\|_2 \leq \sqrt{q|\tilde{h}|} \|\varphi\|_{N,2\delta}^{2/(n-2)} \|\eta f\|_N + \sqrt{q\|h\|_p} \|\eta f\|_{2p/(p-1)} + 2 \|f \nabla \eta\|_2$$

By Sobolev embedding, we know that there exists a positive constant c , which depends only on n , such that

$$\|\eta f\|_N \leq c(\|\eta \nabla f\|_2 + \|f \nabla \eta\|_2 + \|\eta f\|_2)$$

The choice of q ($q < N$) and inequality (12) permit to write

$$(13) \quad (1 - c\sqrt{N|\tilde{h}|} \|\varphi\|_{N,2\delta}^{2/(n-2)}) \|\eta f\|_N \leq c(\sqrt{N\|h\|_p} \|\eta f\|_{2p/(p-1)} + 3 \|f \nabla \eta\|_2 + \|\eta f\|_2)$$

We choose δ sufficiently small such that

$$\|\varphi\|_{N,2\delta}^{2/(n-2)} \leq 1/(2c\sqrt{N|\tilde{h}|})$$

when l goes to $+\infty$, we deduce that there exists a positive constant C , which depends on $n, \delta, \|\eta\|_\infty, \|\nabla\eta\|_\infty, \|h\|_p$ and $|\tilde{h}|$ such that

$$\|\varphi^q\|_{N,2\delta} \leq C(\|\varphi^q\|_2 + \|\varphi^q\|_{2p/(p-1)})$$

$\frac{2p}{p-1}q < N$ and φ is bounded in L^N , hence

$$\|\varphi\|_{qN,2\delta} \leq C$$

If $(\eta_i)_{i \in I}$ is a partition of unity subordinate to the covering $\{B_{P_i}(\delta)\}_{i \in J}$ on M

$$\|\varphi\|_{qN}^{qN} = \sum_{i \in I} \|\eta_i \varphi\|_{qN, \delta_i}^{qN} \leq C$$

Hence $\varphi \in L^{qN}$ with $qN > N$. The remark in the begining of the proof implies the theorem. \square

Proposition 6. Let (M, g) be n -dimensional compact smooth Riemannian manifold. $L := \Delta_g + h$ is a linear operator, with $h \in L^p(M)$ and $p > n/2$. If the smallest eigenvalue λ of L is positive then

i. L est coercive, in other words there exists $c > 0$ such that

$$\forall \psi \in H_1(M) \quad (L\psi, \psi)_{L^2} \geq c(\|\nabla\psi\|_2^2 + \|\psi\|_2^2)$$

ii. The opertor $L : H_2^p(M) \longrightarrow L^p(M)$ is invertible.

Proof. L admits a smallest eigenvalue because if λ is an eigenvalue associated to the eigenfunction ψ then there exists $C > 0$ such that

$$\lambda \|\psi\|_2^2 = (L\psi, \psi)_{L^2} = \|\nabla\psi\|_2^2 + \int_M h\psi^2 dv \geq -\|h\|_p \|\psi\|_{2p/(p-1)}^2 \geq -C\|h\|_p \|\psi\|_2^2$$

Hence $\lambda \geq -C\|h\|_p$. If λ is the smallest eigenvalue of L then

$$\lambda = \inf_{\varphi \in H_1(M) - \{0\}} \frac{E(\varphi)}{\|\varphi\|_2^2}$$

where

$$E(\varphi) = (L\varphi, \varphi)_{L^2} = \int_M |\nabla\varphi|^2 + h\varphi^2 dv$$

So, for any $\varphi \in H_1(M)$

$$(14) \quad E(\varphi) \geq \lambda \|\varphi\|_2^2$$

Suppose that L is non coercive, then there exists a sequence $(\psi_i)_{i \in \mathbb{N}}$ in $H_1(M)$, which satisfies

$$E(\psi_i) < \frac{1}{i}(\|\nabla\psi_i\|_2^2 + \text{vol}(M)^{2/n}) \text{ and } \|\psi_i\|_N = 1$$

It implies

$$(1 - \frac{1}{i})E(\psi_i) < \frac{\text{vol}(M)^{2/n}}{i} - \frac{1}{i} \int_M h\psi_i^2 dv$$

because $|\int_M h\psi_i^2 dv| \leq \|h\|_{n/2} \|\psi_i\|_N^2$, $\lim_{i \rightarrow +\infty} E(\psi_i) \leq 0$. On other hands $E(\psi_i) \geq \lambda \|\psi_i\|_2^2$ with $\lambda > 0$. Which is impossible.

If $L\psi = 0$, then, using (14), $\varphi = 0$. So L is injective.

Let $f \in L^p(M)$. Let us prove that the following equation admit a solution $\psi \in H_2^p(M)$

$$(15) \quad \Delta\varphi + h\varphi = f$$

We minimize the functional E defined in the begining of the proof. Let define μ as follow

$$(16) \quad \mu = \inf\{E(\varphi)/\varphi \in H_1(M), \int_M f\varphi dv = 1\}$$

and $(\psi_i)_{i \in \mathbb{N}}$ a sequence in $H_1(M)$ which minimizes E , then

$$\lim_{i \rightarrow +\infty} E(\psi_i) = \mu \text{ and } \int_M f\psi_i dv = 1$$

Without loss of generalities, we suppose that for any nonnegative integer i , $E(\psi_i) \leq \mu + 1$. It implies

$$c(\|\nabla\psi_i\|_2^2 + \|\psi_i\|_2^2) \leq E(\psi_i) \leq \mu + 1$$

because L is coercive. We conclude that $(\psi_i)_{i \in \mathbb{N}}$ is bounded in $H_1(M)$. The Kondrakov theorem and Banach theorem imply that there exists a subsequence $(\psi_j)_{j \in \mathbb{N}}$ such that

- * $\psi_j \rightharpoonup \psi$ weakly in $H_1(M)$
- * $\psi_j \rightarrow \psi$ strongly in $L^s(M)$ for all $1 \leq s < N$
- * $\psi_j \rightarrow \psi$ almost everywhere.

Then (ψ_j) converge strongly in $L^{2p/(p-1)}(M)$ because $2p/(p-1) < N$. So

$$\int_M f\psi dv = 1 \text{ and } \int_M h\psi_j^2 dv \rightarrow \int_M h\psi^2 dv$$

The weak convergence in $H_1(M)$ and the strong convergence $L^2(M)$ imply

$$\lim_{j \rightarrow +\infty} \|\nabla\psi_j\|_2 \geq \|\nabla\psi\|_2$$

We conclude that $E(\psi) \leq \mu$, hence $E(\psi) = \mu$. If we write Euler-Lagrange equation for ψ , we find that it is a weak solution in $H_1(M)$ of equation (15). It remains to prove that $\psi \in H_2^p(M)$.

Suppose that $\psi \in L^{s_i}(M)$. Then $hu \in L^{\frac{ps_i}{p+s_i}}(M)$, Hence $\Delta u \in L^{\frac{ps_i}{p+s_i}}(M)$. Regularity theorem 3 assures that $u \in H_2^{\frac{ps_i}{p+s_i}}(M)$. We know that $H_2^r(M) \subset L^s(M)$ if $r \leq n/2$ with $s = nr/(n-2r)$, and $H_2^r(M) \subset C^{1-[n/r],\beta}(M)$ if $r > n/2$. These inclusions imply the following results

$$\begin{cases} s_0 = N \\ u \in L^{s_{i+1}}(M) \text{ where } s_{i+1} = \frac{np s_i}{np - (p-2n)s_i} & \text{if } s_i \leq \frac{np}{2p-n} \\ u \in H_2^p(M) & \text{if } s_i > \frac{np}{2p-n} \end{cases}$$

If there exists $i \in \mathbb{N}$ such that $s_i > \frac{np}{2p-n}$, which is equivalent to $\frac{ps_i}{p+s_i} > n/2$ then $u \in C^{0,\beta}(M)$, which implies $\Delta u \in L^p(M)$, hence $u \in H_2^p(M)$. If there exists $i \in \mathbb{N}$ such that $s_i = \frac{np}{2p-n}$ then $u \in L^\infty(M)$ and we conclude by regularity theorem that $u \in H_2^p(M)$. Suppose that for any $i \in \mathbb{N}$, $s_i < \frac{np}{2p-n}$, the sequence $(s_i)_{i \in \mathbb{N}}$ is increasing and bounded from above, it converges to $s = 0$ which is impossible. \square

Theorem 7. *Let (M, g) be a n -dimensional smooth compact Riemannian manifold. For all $\varepsilon > 0$, there exists $A(\varepsilon) > 0$ such that*

$$\forall \varphi \in H_1(M) \quad \|\varphi\|_N \leq (K(n, 2) + \varepsilon) \|\nabla\varphi\|_2 + A(\varepsilon) \|\varphi\|_2$$

where $N = \frac{2n}{n-2}$ and $K(n, 2) = \frac{2}{\sqrt{n(n-2)}} \omega_n^{-1/n}$

The inequality of this theorem is a particular case of a more general one. More further details are given in the Aubin's book [2].

3. EXISTENCE THEOREM

We consider the following equation :

$$(17) \quad \Delta_g \psi + h\psi = \tilde{h}\psi^{\frac{n+2}{n-2}}$$

where $\psi \in H_1(M)$, $h \in L^p(M)$ with $p > n/2$ and \tilde{h} is a reel number. As mentionned in the introduction, this kind of equation are called Yamabe type equation. In the particular case when $h = \frac{n-2}{4(n-1)} R_g$, equation (17) is the Yamabe equation (1). To solve this equation, we use the variational method.

We define the energy E of $\psi \in H_1(M)$ by:

$$(18) \quad E(\psi) = \int_M |\nabla\psi|^2 + h\psi^2 dv$$

and we consider the functional I_g defined for all $\psi \in H_1(M) - \{0\}$ by

$$(19) \quad I_g(\psi) = \frac{E(\psi)}{\|\psi\|_N^2}$$

We denote

$$(20) \quad \mu(g) = \inf_{\psi \in H_1(M) - \{0\}, \psi \geq 0} I_g(\psi) = \inf_{\|\psi\|_N = 1, \psi \geq 0} E(\psi)$$

with $N = \frac{2n}{n-2}$. the main result of this section is

Theorem 8. *If $p > n/2$ and*

$$\mu(g) < K^{-2}(n, 2)$$

then equation (17) admits a positive solution $\varphi \in H_2^p(M) \subset C^{1-[n/p], \beta}(M)$, which minimizes I_g (i.e. $E(\varphi) = \mu(g) = \tilde{h}$ and $\|\varphi\|_N = 1$). where $\beta \in (0, 1)$.

To proof this theorem, we need the following lemma, proven by Brezis and Lieb[4]

Lemma 9. *Let $(f_i)_{i \in \mathbb{N}}$ be a sequence of mesurable functions in (Ω, Σ, μ) . If $(f_i)_{i \in \mathbb{N}}$ is uniformly bounded in L^p with $0 < p < +\infty$ and $f_i \rightarrow f$ almost everywhere, then*

$$\lim_{i \rightarrow +\infty} [\|f_i\|_p^p - \|f_i - f\|_p^p] = \|f\|_p^p$$

Proof of theorem 8. We check that $\mu(g)$ is finite. In fact, using Hölder inequality, we have

$$E(\psi) \geq -\|h\|_{n/2} \|\psi\|_N^2$$

we deduce that $\mu(g) \geq -\|h\|_{n/2} > -\infty$.

Let $(\varphi_i)_{i \in \mathbb{N}}$ be a minimizing sequence:

$$(21) \quad E(\varphi_i) = \mu(g) + o(1), \quad \|\varphi_i\|_N = 1 \text{ et } \varphi_i \geq 0$$

Applying Hölder inequality again for the equation above, we obtain

$$\begin{aligned} \|\nabla \varphi_i\|_2^2 &\leq \|h\|_{n/2} + \mu(g) + o(1) \\ \|\varphi_i\|_2^2 &\leq (\text{vol}(M))^{2/n} \end{aligned}$$

We conclude that $(\varphi_i)_{i \in \mathbb{N}}$ is bounded in $H_1(M)$. Without loss of generalities, we suppose that there exists $\varphi \in H_1(M)$ such that

- *: $\varphi_i \rightharpoonup \varphi$ weakly in $H_1(M)$
- *: $\varphi_i \rightarrow \varphi$ strongly in $L^s(M)$ for any $s \in [1, N)$
- *: $\varphi_i \rightarrow \varphi$ almost everywhere

We deduce that

$$\int_M |h| |\varphi_i - \varphi|^2 dv \leq \|h\|_p \|\varphi_i - \varphi\|_{2p/(p-1)}^2 \rightarrow 0 \text{ strongly because } 2p/(p-1) < N$$

Let $\psi_i = \varphi_i - \varphi$, then $\psi_i \rightarrow 0$ weakly in $H_1(M)$, strongly in $L^q(M)$ for any $q < N$.

We have $\|\nabla \varphi_i\|_2^2 = \|\nabla \psi_i\|_2^2 + \|\nabla \varphi\|_2^2 + 2 \int_M \nabla \psi_i \cdot \nabla \varphi dv$. Hence

$$E(\varphi_i) = E(\varphi) + \|\nabla \psi_i\|_2^2 + o(1)$$

We know that $E(\varphi) \geq \mu(g) \|\varphi\|_N^2$ by definition of $\mu(g)$, and $E(\varphi_i) = \mu(g) + o(1)$ by definition of $(\varphi_i)_{i \in \mathbb{N}}$. We conclude

$$(22) \quad \mu(g) \|\varphi\|_N^2 + \|\nabla \psi_i\|_2^2 \leq \mu(g) + o(1)$$

Using lemma 9 for $(\varphi_i)_{i \in \mathbb{N}}$, we obtain

$$(23) \quad \|\psi_i\|_N^N + \|\varphi\|_N^N + o(1) = 1$$

$$(24) \quad \|\psi_i\|_N^2 + \|\varphi\|_N^2 + o(1) \geq 1$$

Theorem 7 gives

$$\|\psi_i\|_N^2 \leq (K^2(n, 2) + \varepsilon) \|\nabla \psi_i\|_2^2 + o(1)$$

Inequality (24) becomes

$$(K^2(n, 2) + \varepsilon)\|\nabla\psi_i\|_2^2 + \|\varphi\|_N^2 + o(1) \geq 1$$

Using the last inequality in (22), we obtain

$$\mu(g)\|\varphi\|_N^2 + \|\nabla\psi_i\|_2^2 \leq \mu(g)[(K^2(n, 2) + \varepsilon)\|\nabla\psi_i\|_2^2 + \|\varphi\|_N^2] + o(1)$$

Finally

$$[1 - \mu(g)(K^2(n, 2) + \varepsilon)]\|\nabla\psi_i\|_2^2 \leq o(1)$$

If $\mu(g) < K^{-2}(n, 2)$, we can choose ε such that the first factor of this inequality becomes positive. We deduce that $(\psi_i)_{i \in \mathbb{N}}$ converges strongly to zero in $H_1(M)$, $\varphi_i \rightarrow \varphi$ strongly in $H_1(M)$ and $L^N(M)$. Hence $I_g(\varphi) = \mu(g)$.

We have just found a non trivial solution of the following Yamabe type equation

$$\Delta\psi + h\psi = \mu(g)\psi^{N-1}$$

which satisfies $\|\varphi\|_N = 1$ and $\varphi \geq 0$. Theorem 5 implies $\varphi \in H_2^p(M) \subset C^{1-[n/p], \beta}(M)$ and $\varphi > 0$. \square

4. THE CHOICE OF THE METRIC

From now until the end of this paper, M is a compact smooth manifold of dimension $n \geq 3$. Denote by T^*M the cotangent space of M .

Assumption (H): g is a metric in the Sobolev space $H_2^p(M, T^*M \otimes T^*M)$ with $p > n$. There exists a point $P_0 \in M$ and $\delta > 0$ such that g is smooth in the ball $B_{P_0}(\delta)$.

We can suppose that g is C^2 instead of C^∞ in this ball, but it is not an important point.

Actually our objectif, in this section is to study the Yamabe problem when the metric g admits a finite number of points with singularities and smooth out side these points. The assumption (H) generalizes this conditions and define the notion of "singularities".

By Sobolev embedding, $H_2^p(M, T^*M \otimes T^*M) \subset C^{1, \beta}(M, T^*M \otimes T^*M)$ for some $\beta \in (0, 1)$. Hence the metrics which satisfy assumption (H) are $C^{1, \beta}$. The Christoffels belong to $H_1^p \subset C^\beta(M)$. Riemann curvature tensor, Ricci tensor and scalar curvature are in L^p . An example of metric which satisfies assumption (H) is $g = (1 + d(P_0, \cdot)^{2-\alpha})^m g_0$ where g_0 is a smooth metric, $\alpha \in (0, 1)$ and $d(P_0, \cdot)$ is the distance function.

We obtain many results which are true for metrics in $H_2^p(M, T^*M \otimes T^*M)$, with $p > n/2$. In the assumption (H), we add the condition that $p > n$ to have a continuous Christoffels for $g \in H_2^p(M, T^*M \otimes T^*M)$. The assumption (H) is sufficient to prove the Aubin's conjecture 2 (cf. theorem 21), and to construct the Green function of the conformal Laplacian (cf. section 7).

We consider the following problem:

Problem 10. Let g be a metric which satisfies the assumption (H). Does there exist a conformal metric \tilde{g} for which the scalar curvature $R_{\tilde{g}}$ is constant ?

It is clear that if the initial metric g is smooth then the problem above is the Yamabe problem 1, which is completely solved. We will prove that the answer to this problem is positive. The following proposition tell us that the conformal class of the metrics is well defined when the metrics are in H_2^p .

Proposition 11. Let g be a metric in H_2^p and $\psi \in H_2^p(M)$ a positive function. If $p > n/2$ then the metric $\tilde{g} = \psi^{\frac{4}{n-2}}g$ is well defined, and it is in the same space as g .

Proof. Using Sobolev embedding, it is easy to check that $H_2^p(M)$ is an algebra for any $p > n/2$. This proposition is a consequence of this fact. \square

In their paper [6] about the Yamabe problem, Lee and Parker proved that on every compact Riemannian manifold (M, g) , there exist a normal coordinates system $\{(U_i, x_i)\}_{i \in I}$ and metric g' conformal to g such that $\det g' = 1 + O(|x|^m)$ with m as big as we want. Cao [5] and Günther [8] proved that we can get $\det g' = 1$.

Definition 12. \tilde{g} is a Cao–Günther metric if it is conformal to g and there exist a coordinates system such that $\det \tilde{g} = 1$.

Theorem 13 (Cao–Günther). *Let M be $C^{a+2, \beta}$ compact manifold of dimension n with $a \in \mathbb{N}$, $\beta \in (0, 1)$, g be a $C^{a+1, \beta}$ –Riemannian metric, and P be a point in M . Then there exists a $C^{a+1, \beta'}$ –positive function φ with $\beta' \in (0, \beta)$ such that $\det(\varphi g) = 1$ in a normal coordinates system with origin P .*

Notice that if the metric $g \in H_2^p(M, T^*M \otimes T^*M)$ with $p > n$ then it belongs to $C^{1, \beta}$. Hence the manifold (M, g) admits a Cao–Günther metric. It is not really useful to suppose that the metric is smooth in a ball, for the existence of this kind of metrics.

5. CONFORMAL LAPLACIAN

Definition 14. The conformal Laplacian of Riemannian manifold (M, g) is the operator L_g , defined by :

$$L_g = \Delta_g + \frac{n-2}{4(n-1)} R_g$$

It is known that the conformal Laplacian, when g is smooth, is conformally invariant. Actually it verifies (25) strongly. We prove that we have this property even when the metric is in $H_2^p(M, T^*M \otimes T^*M)$.

Proposition 15. $g \in H_2^p(M, T^*M \otimes T^*M)$ is a Riemannian metric on M with $p > n/2$. If $\tilde{g} = \psi^{\frac{4}{n-2}} g$ is a conformal metric to g with $\psi \in H_2^p(M)$ and $\psi > 0$ then L is weakly conformally invariant, which means that

$$(25) \quad \forall u \in H_1(M) \quad \psi^{\frac{n+2}{n-2}} L_{\tilde{g}}(u) = L_g(\psi u) \quad \text{weakly}$$

Moreover if $\mu(g) > 0$ then the conformal Laplacian $L_g = \Delta_g + \frac{n-2}{4(n-1)} R_g$ is invertible and coercive.

Proof. Recall that $dv_{\tilde{g}} = \psi^{\frac{2n}{n-2}} dv$ and

$$\forall u, w \in L^2(M) \quad (u, w)_{g, L^2} = \int_M u w dv_g$$

is the scalar product in $L^2(M)$ with the metric g .

For all $u, w \in H_1(M)$:

$$\begin{aligned} (\psi^{\frac{2n}{n-2}} L_{\tilde{g}} u, w)_{g, L^2} &= (L_{\tilde{g}} u, w)_{\tilde{g}, L^2} \\ &= \int_M \tilde{g}(\nabla u, \nabla w) + \frac{n-2}{4(n-1)} R_{\tilde{g}} u w dv_{\tilde{g}} \\ &= \int_M \psi^2 g(\nabla u, \nabla w) + \frac{n-2}{4(n-1)} R_{\tilde{g}} \psi^{\frac{n+2}{n-2}} (u w \psi) dv_g \end{aligned}$$

We know that the scalar curvatures R_g and $R_{\tilde{g}}$ are related by Yamabe equation (1), which is equivalent to

$$L_g \psi = \frac{n-2}{4(n-1)} R_{\tilde{g}} \psi^{\frac{n+2}{n-2}} \quad \text{weakly}$$

then

$$(L_g \psi, u w \psi)_{g, L^2} = \frac{n-2}{4(n-1)} (R_{\tilde{g}} \psi^{\frac{n+2}{n-2}}, u w \psi)_{g, L^2}$$

Hence

$$\begin{aligned}
(\psi^{\frac{2n}{n-2}} L_{\tilde{g}} u, w)_{g, L^2} &= \int_M \psi^2 g(\nabla u, \nabla w) + g(\nabla \psi, \nabla(uw\psi)) + \frac{n-2}{4(n-1)} R_g \psi(uw\psi) dv_g \\
(26) \quad &= \int_M g(\nabla(\psi u), \nabla(w\psi)) + \frac{n-2}{4(n-1)} R_g(\psi u)(w\psi) dv_g \\
&= (\psi L_g(\psi u), w)_{g, L^2}
\end{aligned}$$

We used the fact that $u\psi$ and $w\psi$ belong to $H_1(M)$, indeed we have the the following Sobolev embedding

$$H_2^p(M) \subset C^{1-[n/p], \beta}(M), \quad H_1^p(M) \subset L^{\frac{pn}{n-p}}(M) \text{ and } H_1(M) \subset L^{\frac{2n}{n-2}}(M)$$

Let us prove that L_g is invertible and coercive. Let λ be the smallest eigenvalue of L_g with positive eigenfunction $\varphi \in H_1(M)$, then

$$\lambda \|\varphi\|_2^2 = (L_g \varphi, \varphi)_{g, L^2} = I_g(\varphi) \|\varphi\|_N^2 \geq \mu(g) \|\varphi\|_N^2 > 0$$

hence $\lambda > 0$. We conclude the result, by applying proposition 6. \square

6. YAMABE CONFORMAL INVARIANT

In the case of smooth metrics, $\mu(g)$ is conformally invariant, which means that if g and \tilde{g} are two smooth conformal metrics then $\mu(g) = \mu(\tilde{g})$. The next proposition shows that we can extend this property to metrics in H_2^p .

Proposition 16. Let M be a smooth compact manifold of dimension $n \geq 3$. Let g and $\tilde{g} = \psi^{\frac{4}{n-2}} g$ be two metrics in H_2^p , with $\psi \in H_2^p(M)$ positive. if $p > n/2$ then

$$\mu(g) = \mu(\tilde{g})$$

Proof. Let $u \in H_1(M)$ be test function for the Yamabe functional I_g . Notice that $E(u) = (L_g(u), u)_{g, L^2}$. then

$$I_{\tilde{g}}(u) = (L_{\tilde{g}}(u), u)_{\tilde{g}, L^2} \|u\psi\|_N^{-2}$$

Using proposition 15, we deduce that

$$I_{\tilde{g}}(u) = (L_g(\psi u), \psi u)_{g, L^2} \|u\psi\|_N^{-2}$$

Finally

$$(27) \quad I_{\tilde{g}}(u) = I_g(\psi u)$$

Which implies that $\mu(g) = \mu(\tilde{g})$. So this invariant depends only on the conformal class $[g]$ and the manifold M . \square

7. GREEN FUNCTION

Definition 17. Let (M, g) be a compact Riemannian manifold and P be a point in M . We call G_P the Green function on P of the linear operator L , if it satisfies

$$LG_P = \delta_P (\iff \forall f \in C^\infty(M) \quad \langle G_P, Lf \rangle = f(P))$$

Proposition 18 shows the existence of such function for the operator $L = \Delta + h$ with a positive continuous function h . Unfortunately, the method used to construct this function doesn't work when h belongs to $L^p(M)$. This case holds for the conformal Laplacian operator L_g , because $R_g \in L^p(M)$. But, using proposition 19, we construct this function and we obtain corollary 20.

Proposition 18. Let h be a positive continuous and $P \in M$. g is a metric satisfying assumption (H). There exists a unique Green function G_P for the operator $L = \Delta_g + h$ which satisfies $LG_P = \delta_P$ and

- (i) G_P is smooth in $B_{P_0}(\delta) - \{P\}$
- (ii) $G_P \in C^2(M - \{P\})$

(iii) There exists $c > 0$ such that for any $Q \in M - \{P\}$, $|G_P(Q)| \leq cd(P, Q)^{2-n}$

Proof. G_P is unique because L is invertible. In fact, if λ is an eigenvalue of L and φ is a positive eigenfunction associated to λ then

$$\lambda \|\varphi\|_2^2 = (L\varphi, \varphi)_{L^2} = E(\varphi) > 0$$

Hence $\lambda > 0$. To conclude, it is sufficient to apply proposition 6. For the existence of such function, we follow Aubin's [2] construction for the Laplacian, in the case of smooth metrics. We choose a decreasing positive smooth radial function $f(r)$, equal to 1 for $r < \delta/2$ and zero for $r \geq \delta(M)$ the injectivity radius of M . We define the following functions

$$\begin{aligned} H(P, Q) &= \frac{f(r)}{(n-2)\omega_{n-1}} r^{2-n} \text{ with } r = d(P, Q) \\ \Gamma^1(P, Q) &= -L_Q H(P, Q) \\ \forall i \in \mathbb{N}^* \quad \Gamma^{i+1}(P, Q) &= \int_M \Gamma^i(P, S) \Gamma^1(S, Q) dv(S) \end{aligned}$$

Then

$$|\Gamma^1(P, Q)| \leq cd(P, Q)^{2-n}$$

We show that

$$\forall i \geq 1 \quad |\Gamma^i(P, Q)| \leq \begin{cases} cd(P, Q)^{2i-n} & \text{if } 2i < n \\ c(1 + \log d(P, Q)) & \text{if } 2i = n \\ c & \text{if } 2i > n \end{cases}$$

In the last case Γ^i is continuous.

More details are given in Aubin' book [2].

The Green function of L is given by

$$(28) \quad G_P(Q) = H(P, Q) + \sum_{i=1}^k \int_M \Gamma^i(P, S) H(S, Q) dv(S) + F_P(Q)$$

where F_P satisfies

$$LF_P = \Gamma^{k+1}(P, \cdot)$$

We choose $k = [n/2]$, $\Gamma^{k+1}(P, \cdot)$ is continuous. Regularity theorem 3 implies that F_P is C^2 .

(i) $L_g G_P = 0$ in $B_{P_0}(\delta) - \{P\}$ and the metric is smooth on $B_{P_0}(\delta)$, regularity theorem assure that G_P is smooth on $B_{P_0}(\delta) - \{P\}$, with $P \in M$.

(ii) We have also $LG_P = 0$ in $M - \{P\}$. We conclude that G_P is C^2 in $M - \{P\}$.

(iii) In the expression (28), we notice that the leading term, in the neighborhood of P , is $H(P, Q)$, then for all $P \neq Q$,

$$|G_P(Q)| \leq cd(P, Q)^{2-n}$$

□

Proposition 19. Let g be a metric in $H_2^p(M, T^*M \otimes T^*M)$, $\tilde{g} = \psi^{\frac{4}{n-2}} g$ is conformal to g with $\psi \in H_2^p(M)$ positive and $p > n/2$. We suppose that $L_{\tilde{g}}$ admits a Green function on P , denoted \tilde{G}_P , then L_g admits a Green function, denoted G_P and it is given by

$$\forall Q \in M - \{P\} \quad G_P(Q) = \psi(P)\psi(Q)\tilde{G}_P(Q)$$

Proof. For any function $\varphi \in C^\infty(M)$:

$$\begin{aligned}
\langle \psi(P) \psi \tilde{G}_P, L_g \varphi \rangle_g &= \psi(P) \int_M \tilde{G}_P \psi L_g [\psi(\frac{\varphi}{\psi})] dv_g \\
&= \psi(P) \int_M \tilde{G}_P L_{\tilde{g}} \frac{\varphi}{\psi} dv_{\tilde{g}} \\
&= \psi(P) \langle \tilde{G}_P, L_{\tilde{g}} \frac{\varphi}{\psi} \rangle_{\tilde{g}} \\
&= \varphi(P)
\end{aligned}$$

The second equality above is obtained by the weak conformal invariance of the conformal Laplacian (see proposition 15). We know that for any $Q \in M - \{P\}$

$$|\tilde{G}_P(Q)| \leq cd(P, Q)^{2-n}$$

then $G_P \in L^s(M)$, for any $s \in [1, n/(n-2))$ and $L_{\tilde{g}} \frac{\varphi}{\psi} \in L^p(M)$ with $p > n/2$. We choose s such that $\langle \tilde{G}_P, L_{\tilde{g}} \frac{\varphi}{\psi} \rangle_{\tilde{g}}$ is finite. Hence the third equality is well defined. \square

Corollary 20. *g is a Riemannian metric, satisfying assumption (H). If $\mu(g) > 0$ then the conformal Laplacian L_g admits a Green function G_{P_0} which satisfies $LG_{P_0} = \delta_{P_0}$ and*

- (i) G_{P_0} is smooth in $B_{P_0}(\delta) - \{P_0\}$
- (ii) $G_{P_0} \in H_2^p(M - B_{P_0}(r))$ for any $r > 0$.
- (iii) There exists $c > 0$ such that for any $Q \in B_{P_0}(\delta) - \{P_0\}$, $|G_{P_0}(Q)| \leq cd(P_0, Q)^{2-n}$

Proof. $\mu(g) > 0$, L_g is invertible. We deduce that L_g admits a unique Green function. Using standard variational method (see the proof of proposition 6), we can show that the equation

$$(29) \quad \Delta_g \psi + \frac{n-2}{4(n-1)} R_g \psi = \mu_{q,G}(g) \psi^{q-1}$$

admits a positive solution $\psi \in H_2^p(M)$ when $2 \leq q < N$, with

$$\mu_{q,G}(g) = \inf_{\psi \in H_1(M) - \{0\}} \frac{E(\psi)}{\|\psi\|_q^2}$$

Moreover, g is smooth in $B_{P_0}(\delta)$, regularity theorem shows that ψ is also smooth in the same ball. The metric $\tilde{g} := \psi^{\frac{4}{n-2}} g$ satisfies assumption (H). Using Yamabe equation (1), we deduce that the scalar curvature of \tilde{g} is

$$R_{\tilde{g}} = \frac{4(n-1)}{n-2} \mu_{q,G}(g) \psi^{q-N}$$

Hence $R_{\tilde{g}}$ is positive continuous because $\mu_{q,G}(g) > 0$. Now, we are able to use proposition 18, which assure the existence of the Green function \tilde{G}_{P_0} for $L_{\tilde{g}}$ with the metric \tilde{g} . using proposition 19, we conclude that $G_{P_0} = \psi(P_0) \psi \tilde{G}_{P_0}$ is the Green function of L_g . The metrics g and \tilde{g} are smooth in $B_{P_0}(\delta)$ and \tilde{G}_{P_0} satisfies the properties of proposition 18, then the properties announced for G_{P_0} are valid. \square

8. EXISTENCE THEOREM

Theorem 21. *Let M be a smooth compact manifold of dimension $n \geq 3$, g is a Riemannian metric which satisfies the assumption (H). If (M, g) is not conformal to the sphere (S_n, g_{can}) then $\mu(g) < K^{-2}(n, 2)$.*

This theorem assure that Aubin's conjecture 2 still valid for any metric satisfying the assumption (H).

To prove this theorem, we use the results of Aubin and Schoen, when the metric g is smooth. The strategy is the following: we construct a test function for the functional I_g , with a support in small geodesic ball. Then the problem is local. We know that the metric g is smooth in $B_{P_0}(\delta)$, so the proof of this theorem is the same as when the metric is smooth everywhere (this

is the point where we need the assumption : g is smooth in $B_{P_0}(\delta)$). After, we consider Aubin and Schoen's test functions.

We need also the following result obtained by Aubin [3], for the Green function of L_g :

Theorem 22. *If g is a Cao–Günther metric, L_g is invertible and the normalized Green function G_{P_0} have the following expression*

$$G_{P_0}(Q) = r^{2-n} + A + O(r)$$

in a neighborhood of P_0 with $r = d(P_0, Q)$, then $A > 0$, except if (M, g) is conformal to (S_n, g_{can}) for which $A = 0$.

Proof of theorem 21. If $\mu(g) \leq 0$ then the inequality is obvious. From now until the end of the proof, we suppose that $\mu(g) > 0$. without loss of generalities, we suppose that g is a Cao–Günther metric given in theorem 13. In fact, $\mu(g)$ is conformally invariant (see proposition 15).

There are two cases which can happen :

(a) The case (M, g) is not conformally flat in a neighborhood of P_0 and $n \geq 6$. We define $\varphi_\varepsilon = \eta v_\varepsilon$, η is a cut-off function with support in $B_{P_0}(2\varepsilon)$, $\eta = 1$ in $B_{P_0}(\varepsilon)$, $2\varepsilon < \delta$ and

$$v_\varepsilon(Q) = \left(\frac{\varepsilon}{r^2 + \varepsilon^2} \right)^{\frac{n-2}{2}} \quad r = d(P_0, Q)$$

$\text{supp} \varphi \subset B_{P_0}(\delta)$ and the metric g is smooth in this ball, we obtain the following lemma (see Aubin [1]):

Lemma 23.

$$\mu(g) \leq I_g(\varphi_\varepsilon) \leq \begin{cases} K^{-2}(n, 2) - c|W_g(P_0)|^2 \varepsilon^4 + o(\varepsilon^4) & \text{si } n > 6 \\ K^{-2}(n, 2) - c|W_g(P_0)|^2 \varepsilon^4 \log \frac{1}{\varepsilon} + O(\varepsilon^4) & \text{si } n = 6 \end{cases}$$

where $|W_g(P_0)|$ is the norm of the Weyl tensor on P_0 .

(Lee et Parker [6] gave a simple proof of this lemma, using the conformal normal coordinates on P_0). Using this lemma, we conclude that $\mu(g) < K^{-2}(n, 2)$.

(b) The case (M, g) is conformally flat in a neighborhood of P_0 or $n = 3, 4$ or 5 . In this coordinates system, the Taylor expansion of the Green function is:

$$G_{P_0}(Q) = r^{2-n} + A + O(r)$$

with $r = d(P_0, Q)$ (see Lee and Parker's paper [6] for the proof of this expansion).

If g satisfies assumption (H) and (M, g) is not conformal to (S_n, g_{can}) , then theorem 22 assure that $A > 0$. Hence we can consider Shoen's test function φ_ε , defined for any $Q \in M$ by:

$$\varphi_\varepsilon(Q) = \begin{cases} v_\varepsilon(Q) & \text{if } Q \in B_{P_0}(\rho_0) \\ \varepsilon_0[G_{P_0} - \eta(G_{P_0} - r^{2-n} - A)](Q) & \text{if } Q \in B_{P_0}(2\rho_0) - B_{P_0}(\rho_0) \\ \varepsilon_0 G_{P_0}(Q) & \text{if } Q \in M - B_{P_0}(2\rho_0) \end{cases}$$

with $2\rho_0 < \delta$, $(\frac{\varepsilon}{\rho_0^2 + \varepsilon^2})^{(n-2)/2} = \varepsilon_0(\rho_0^{2-n} + A)$ and η is a smooth nonnegative decreasing function on \mathbb{R}_+ , with support in $(-2\rho_0, 2\rho_0)$, equal to 1 in $[0, \rho_0]$, the gradient $|\nabla \eta(r)| \leq \rho_0^{-1}$. g is smooth in $B_{P_0}(2\rho_0) \subset B_{P_0}(\delta)$ and $G_{P_0} \in H_2^p(M - B_{P_0}(\rho_0))$ (see corollary 20), then we have the estimate of $\mu(g)$, obtained by Schoen[9]:

Lemma 24.

$$\mu(g) \leq I_g(\varphi_\varepsilon) \leq K^{-2}(n, 2) + c\varepsilon_0^2(c\rho_0 - A)$$

The fact that $A > 0$ allows us to choose ρ_0 sufficiently small ($c\rho_0 < A$) such that $\mu(g) < K^{-2}(n, 2)$.

□

Now, we can state the main theorem which solves the problem 10 for any metric which satisfies assumption (H).

Theorem 25. *Let M be a smooth compact manifold of dimension $n \geq 3$ and g be a metric satisfying assumption (H). There exists a metric \tilde{g} conformal to g such that the scalar curvature $R_{\tilde{g}}$ is constant everywhere. This metric solves the problem 10.*

It means that we can always solve the equation of type Yamabe (17) when $h = \frac{n-2}{4(n-1)}R_g$.

Proof. If (M, g) is conformal to (S_n, g_{can}) then the result is obvious because the scalar curvature of (S_n, g_{can}) is constant. Otherwise (M_n, g) is not conformal to (S_n, g_{can}) . In this case, we have the inequality

$$\mu(g) < K^{-2}(n, 2)$$

given by theorem 21. Using theorem 8, we get a positive solution $\psi \in H_2^p(M)$ of (17), where $h = \frac{n-2}{4(n-1)}R_g$ and $\tilde{h} = \mu(g)$. Using Yamabe equation (1), we deduce that the metric $\tilde{g} = \psi^{\frac{4}{n-2}}g$ has a constant scalar curvature $R_{\tilde{g}} = \frac{4(n-1)}{n-2}\mu(g)$. \square

9. UNIQUENESS OF SOLUTIONS

When the metrics are smooth, if $\mu(g)$ is nonpositive then the solutions of the Yamabe equation (1) are proportional. The following theorem generalizes the uniqueness theorem in the singular case.

Theorem 26. *Let g be a metric in $H_2^p(M, T^*M \otimes T^*M)$, with $p > n$. If $\mu(g) \leq 0$ then the solutions of (1) are proportional.*

Proof. Let φ_1 and φ_2 two positive solutions of (1). The metrics $g_i = \varphi_i^{\frac{4}{n-2}}g$ have a constant scalar curvatures R_i , where $i = 1$ or 2 . Define $\psi = \frac{\varphi_1}{\varphi_2}$, then $g_1 = \psi^{\frac{4}{n-2}}g_2$. It implies that ψ satisfies

$$(30) \quad \Delta_{g_2}\psi + \frac{n-2}{4(n-1)}R_2\psi = \frac{n-2}{4(n-1)}R_1\psi^{\frac{n+2}{n-2}}$$

By regularity theorem 3, we deduce that ψ is $C^{2,\beta}$ because the coefficient of the Laplacian are C^0 . In fact, in a local coordinates system :

$$\Delta_g\psi = -\nabla_i\nabla^i\psi = -g^{ij}(\partial_{ij}\psi - \Gamma_{ij}^k\partial_k\psi)$$

and the Christoffels are in $H_1^p(M)$ then continuous if $p > n$. In other hands, notice that R_1, R_2 have the same sign. Hence, if $\mu(g) < 0$ then $R_i < 0$ for $i = 1$ and 2 . Let $Q_1 \in M$ (resp. $Q_2 \in M$) be a point for which ψ is maximal (resp. minimal). Then $\Delta_{g_2}\psi(Q_1) \geq 0$ and $\Delta_{g_2}\psi(Q_2) \leq 0$. Hence, if we evaluate equation (30) at Q_1 and Q_2 , we obtain :

$$\psi^{\frac{4}{n-2}}(Q_1) \leq \frac{R_2}{R_1} \text{ and } \psi^{\frac{4}{n-2}}(Q_2) \geq \frac{R_2}{R_1}$$

We conclude that $\psi = \frac{R_2}{R_1}$, φ_1 and φ_2 are proportional.

If $\mu(g) = 0$ then $R_1 = R_2 = 0$ and (30) becomes $\Delta_{g_2}\psi = 0$, hence ψ is constant. \square

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